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# SOLVING THE SIMPLEST INTEGRAL EQUATION FORMS EXACTLY ON LIMIT RECURRENCE WITH APPLIED THEIR BEING CONTRACTIVE MAPPINGS

***There have been proved 17 theorems on solving exactly the simplest integral equation forms of the operator mapping $\mathscr{A}$ as the equation $x(t)=\mathscr{A}\left[x(t)\right]$, being contractive. The proved on limit recurrence unique solutions have been displayed in the pre-conclusion table.***

***Доказано 17 теорем по точному решению простейших форм интегрального уравнения от операторного отображения $\mathscr{A}$ как уравнения $x(t)=\mathscr{A}\left[x(t)\right]$, являющимся сжимающим. Доказанные с использованием граничной рекурсии решения отражены в таблице перед заключением.***

***Key words: operator mapping, contractive mapping, metric, metric set, metric space, functional space, integral equation, limit recurrence, approximate solution, unique fixed point, unique solution, mapping kernel, zeroth approximation, the $n$-th approximation, geometrical progression sum, $[0;\,1]$-defined measurable functions.***

## Paper specification

There is a broad nature scope of applying the operator equations, which solutions are found mostly in numerical ways. Some narrowed class of these equations is concluded within the operator mapping $\mathscr{A}$ of a metric set $\boldsymbol{X}_{\mathscr{M}}$ into itself:

$$x(t)=\mathscr{A}\left[x(t)\right] \quad \forall\, x(t)\in \boldsymbol{X}_{\mathscr{M}} \text{ by } t\in \mathscr{M}\subset\mathbb{R}\,, \tag{1}$$

where $\mathscr{M}\subset\mathbb{R}$ is supposed to be a measurable subset of the real axis [1, 2]. Operator equations of the type (1) with respect to the variable $x(t)\in\boldsymbol{X}_{\mathscr{M}}$ may be solved into $x^{*}(t)\in\boldsymbol{X}_{\mathscr{M}}$ numerically, if just the operator $\mathscr{A}$ is a contractive mapping, that is

$$\rho_{\boldsymbol{X}_{\mathscr{M}}}\left(\mathscr{A}\left[x_1(t)\right],\, \mathscr{A}\left[x_2(t)\right]\right)\leqslant \alpha\rho_{\boldsymbol{X}_{\mathscr{M}}}\left(x_1(t),\, x_2(t)\right) \tag{2}$$

for the existing $\alpha\in(0;\,1)$ on the metric $\rho_{\boldsymbol{X}_{\mathscr{M}}}$ $\forall\, x_1(t)\in\boldsymbol{X}_{\mathscr{M}}$ and $\forall\, x_2(t)\in\boldsymbol{X}_{\mathscr{M}}$ as $\mathscr{A}\left[x_1(t)\right]\in\boldsymbol{X}_{\mathscr{M}}$ and $\mathscr{A}\left[x_2(t)\right]\in\boldsymbol{X}_{\mathscr{M}}$ also [3 — 6].

Though the particularity of the operator mapping (1) in the integral equation [7, 8]

$$x(t)=f(t)+\int_0^1 K(t,\, s)\,x(s)\,ds \tag{3}$$

form by the unknown function $x(t)\in\boldsymbol{X}_{[0;\,1]}$, known function $f(t)\in\boldsymbol{X}_{[0;\,1]}$ and the contractive mapping known kernel $K(t,\, s)\in\boldsymbol{X}_{[0;\,1]}\times\boldsymbol{X}_{[0;\,1]}$, solution $x^{*}(t)$ of (3) has a great many of practiced branches. And finding it for some highlighted cases of the known function $f(t)\in\boldsymbol{X}_{[0;\,1]}$ and the kernel $K(t,\, s)\in\boldsymbol{X}_{[0;\,1]}\times\boldsymbol{X}_{[0;\,1]}$ is going to be papered.

## Analysis of the antecedent fundamentals

Assume that the mapping of the space $\boldsymbol{X}_{[0;\,1]}$ into itself in the right member of (3) is contractive. Then, taking $x_0(t)\in\boldsymbol{X}_{[0;\,1]}$ into recurrence

$$x_n(t)=f(t)+\int_0^1 K(t,\, s)\,x_{n-1}(s)\,ds \tag{4}$$

by $n\in\mathbb{N}$ due to the principle of contracting mappings, will get the approximate solution $x^{*}(t)\approx x_n(t)$ for as high $n\in\mathbb{N}$ as possible or acceptable. Besides, this solution is exact for the existing limit

$$x^*(t)=\lim_{n\to\infty}x_n(t)=\lim_{n\to\infty}\left(f(t)+\int_0^1 K(t,s)x_{n-1}(s)ds\right). \tag{5}$$

Certainly, it is impossible to solve (3) as (5) for the kernel $K(t,s)\in\boldsymbol{X}_{[0;1]}\times\boldsymbol{X}_{[0;1]}$ general type. But in some specified cases of such kernel and function $f(t)\in\boldsymbol{X}_{[0;1]}$ there is the opportunity to find the exact solution (5).

### Investigation goal

The being initiated paper goal is to substantiate the exact solution (5) in the following simplest integral equations (3):

$$x(t)=a+b\int_0^1 x(s)ds\,, \tag{6}$$

$$x(t)=at+b\int_0^1 x(s)ds\,, \tag{7}$$

$$x(t)=a+b\int_0^1 tx(s)ds\,, \tag{8}$$

$$x(t)=at+b\int_0^1 tx(s)ds\,, \tag{9}$$

$$x(t)=a+b\int_0^1 sx(s)ds\,, \tag{10}$$

$$x(t)=at+b\int_0^1 sx(s)ds\,, \tag{11}$$

$$x(t)=a+b\int_0^1 tsx(s)ds\,, \tag{12}$$

$$x(t)=at+b\int_0^1 tsx(s)ds\,, \tag{13}$$

whereupon there ought to be solved the generalized (6), (8) as

$$x(t)=a+b\int_0^1 t^q x(s)ds \quad \text{by} \quad q\geqslant 0\,, \tag{14}$$

the generalized (7), (9) as

$$x(t)=at+b\int_0^1 t^q x(s)ds \quad \text{by} \quad q\geqslant 0\,, \tag{15}$$

the generalized (6), (10) as

$$x(t)=a+b\int_0^1 s^m x(s)ds \quad \text{by} \quad m\geqslant 0\,, \tag{16}$$

the generalized (7), (11) as

$$x(t) = at + b\int_0^1 s^m x(s)ds \quad \text{by} \quad m \geqslant 0\,, \tag{17}$$

the generalized (8), (12) as

$$x(t) = a + b\int_0^1 ts^m x(s)ds \quad \text{by} \quad m \geqslant 0\,, \tag{18}$$

the generalized (9), (13) as

$$x(t) = at + b\int_0^1 ts^m x(s)ds \quad \text{by} \quad m \geqslant 0\,, \tag{19}$$

the generalized (18) as

$$x(t) = a + b\int_0^1 t^q s^m x(s)ds \quad \text{by} \quad q \geqslant 0\,,\ m \geqslant 0\,, \tag{20}$$

the generalized (19) as

$$x(t) = at + b\int_0^1 t^q s^m x(s)ds \quad \text{by} \quad q \geqslant 0\,,\ m \geqslant 0\,, \tag{21}$$

and the generalized (21) globally in polynomializing $f(t)$ part as

$$x(t) = \sum_{i=0}^{p} a_i t^i + b\int_0^1 t^q s^m x(s)ds \quad \text{by} \quad p \in \mathbb{N} \cup \{0\}\,,\ q \geqslant 0\,,\ m \geqslant 0\,. \tag{22}$$

Actually, the equation (22) generalizes equations (6) — (21). The substantiation should include bounding the number parameters $a \in \mathbb{R}$, $a_i \in \mathbb{R}\ \ \forall\, i = \overline{0,\, p}$ and $b \in \mathbb{R}$ of the function $f(t) \in \boldsymbol{X}_{[0;\,1]}$ and the kernel $K(t,\, s) \in \boldsymbol{X}_{[0;\,1]} \times \boldsymbol{X}_{[0;\,1]}$ correspondingly.

### Solving the integral equation (6)

**Theorem 1.** The mapping in (6) has the unique fixed point $x^*(t) \in \boldsymbol{X}_{[0;\,1]}$, that is the solution

$$x^*(t) = \frac{a}{1-b} \tag{23}$$

of this equation by $a \in \mathbb{R} \setminus \{0\}$ and $|b| < 1$.

**Proof.** Even not assured in contractiveness of the mapping in (6), there are no restrictions on applying the recurrence (4). Take the zeroth approximation

$$x_0(t) = 0 \quad \forall\, t \in [0;\,1] \tag{24}$$

and will get

$$x_1(t) = a + b\int_0^1 x_0(s)ds = a + b\int_0^1 0ds = a\,, \tag{25}$$

$$x_2(t) = a + b\int_0^1 x_1(s)ds = a + b\int_0^1 ads = a + ab\,, \tag{26}$$

$$x_3(t) = a + b\int_0^1 x_2(s)ds = a + b\int_0^1 (a + ab)ds = a + ab + ab^2, \tag{27}$$

$$x_4(t) = a + b\int_0^1 x_3(s)ds = a + b\int_0^1 \left(a + ab + ab^2\right)ds = a + ab + ab^2 + ab^3. \tag{28}$$

So, for $n \in \mathbb{N}\setminus\{1\}$ the $n$-th approximation

$$x_n(t) = a + b\int_0^1 x_{n-1}(s)ds = a + b\int_0^1 \left(a + \sum_{k=1}^{n-2} ab^k\right)ds = a + \sum_{k=1}^{n-1} ab^k, \tag{29}$$

giving the geometrical progression in the last member. This progression is summable by $|b| < 1$ and then

$$x^*(t) = \lim_{n\to\infty} x_n(t) = \lim_{n\to\infty}\left(a + \sum_{k=1}^{n-1} ab^k\right) = a\lim_{n\to\infty}\left(1 + \sum_{k=1}^{n-1} b^k\right) = \frac{a}{1-b}. \tag{30}$$

For avoiding triviality, the parameter $a \neq 0$, that is $a \in \mathbb{R}\setminus\{0\}$. The theorem has been proved.

**Solving the integral equation (7)**

**Theorem 2.** The mapping in (7) has the unique fixed point $x^*(t) \in \boldsymbol{X}_{[0;\,1]}$, that is the solution

$$x^*(t) = at + \frac{ab}{2(1-b)} \tag{31}$$

of this equation by $a \in \mathbb{R}\setminus\{0\}$ and $|b| < 1$.

**Proof.** Not restricted on applying the recurrence (4), take the zeroth approximation (24) and get

$$x_1(t) = at + b\int_0^1 x_0(s)ds = at + b\int_0^1 0ds = at, \tag{32}$$

$$x_2(t) = at + b\int_0^1 x_1(s)ds = at + b\int_0^1 asds = at + ab\left.\frac{s^2}{2}\right|_0^1 = at + \frac{ab}{2}, \tag{33}$$

$$x_3(t) = at + b\int_0^1 x_2(s)ds = at + b\int_0^1\left(as + \frac{ab}{2}\right)ds = at + ab\left.\frac{s^2}{2}\right|_0^1 + \frac{ab^2}{2} = at + \frac{ab}{2} + \frac{ab^2}{2}, \tag{34}$$

$$x_4(t) = at + b\int_0^1 x_3(s)ds = at + b\int_0^1\left(as + \frac{ab}{2} + \frac{ab^2}{2}\right)ds = at + ab\left.\frac{s^2}{2}\right|_0^1 + \frac{ab^2}{2} + \frac{ab^3}{2} = at + \frac{ab}{2} + \frac{ab^2}{2} + \frac{ab^3}{2}. \tag{35}$$

So, for $n \in \mathbb{N}\setminus\{1\}$ the $n$-th approximation

$$\begin{aligned} x_n(t) &= at + b\int_0^1 x_{n-1}(s)ds = at + b\int_0^1\left(as + \frac{a}{2}\sum_{k=1}^{n-2} b^k\right)ds = at + ab\left.\frac{s^2}{2}\right|_0^1 + \frac{ab}{2}\sum_{k=1}^{n-2} b^k = \\ &= at + \frac{ab}{2} + \frac{ab}{2}\sum_{k=1}^{n-2} b^k = at + \frac{ab}{2}\left(1 + \sum_{k=1}^{n-2} b^k\right), \end{aligned} \tag{36}$$

giving the geometrical progression in the last member sum. This progression is summable by $|b| < 1$ and then

$$x^*(t) = \lim_{n\to\infty} x_n(t) = \lim_{n\to\infty}\left[at + \frac{ab}{2}\left(1 + \sum_{k=1}^{n-2} b^k\right)\right] =$$

$$= at + \frac{ab}{2}\lim_{n\to\infty}\left(1 + \sum_{k=1}^{n-2} b^k\right) = at + \frac{ab}{2}\cdot\frac{1}{1-b} = at + \frac{ab}{2(1-b)}. \tag{37}$$

For avoiding triviality, the parameter $a \in \mathbb{R}\setminus\{0\}$. The theorem has been proved.

**Solving the integral equation (8)**

**Theorem 3.** The mapping in (8) has the unique fixed point $x^*(t) \in \boldsymbol{X}_{[0;\,1]}$, that is the solution

$$x^*(t) = a + \frac{2ab}{2-b}t \tag{38}$$

of this equation by $a \in \mathbb{R}\setminus\{0\}$ and $|b| < 2$.

**Proof.** Not restricted on applying the recurrence (4), take the zeroth approximation (24) and get

$$x_1(t) = a + b\int_0^1 tx_0(s)\,ds = a + b\int_0^1 t\cdot 0\,ds = a, \tag{39}$$

$$x_2(t) = a + b\int_0^1 tx_1(s)\,ds = a + b\int_0^1 ta\,ds = a + abt, \tag{40}$$

$$x_3(t) = a + b\int_0^1 tx_2(s)\,ds = a + b\int_0^1 t(a + abs)\,ds = a + abt + ab^2t\left.\frac{s^2}{2}\right|_0^1 = a + abt + \frac{ab^2}{2}t, \tag{41}$$

$$x_4(t) = a + b\int_0^1 tx_3(s)\,ds = a + b\int_0^1 t\left(a + abs + \frac{ab^2}{2}s\right)ds =$$

$$= a + abt + ab^2t\left.\frac{s^2}{2}\right|_0^1 + ab^3t\left.\frac{s^2}{4}\right|_0^1 = a + abt + \frac{ab^2}{2}t + \frac{ab^3}{4}t. \tag{42}$$

So, for $n \in \mathbb{N}\setminus\{1\}$ the $n$-th approximation

$$x_n(t) = a + b\int_0^1 tx_{n-1}(s)\,ds = a + b\int_0^1 t\left(a + \sum_{k=1}^{n-2}\frac{ab^k s}{2^{k-1}}\right)ds = a + abt + abt\sum_{k=1}^{n-2}\frac{b^k}{2^{k-1}}\cdot\left.\frac{s^2}{2}\right|_0^1 =$$

$$= a + abt + abt\sum_{k=1}^{n-2}\frac{b^k}{2^k} = a + abt\left(1 + \sum_{k=1}^{n-2}\frac{b^k}{2^k}\right), \tag{43}$$

giving the geometrical progression in the last member sum. This progression is summable by $|b| < 2$ and then

$$x^*(t) = \lim_{n\to\infty} x_n(t) = \lim_{n\to\infty}\left[a + abt\left(1 + \sum_{k=1}^{n-2}\frac{b^k}{2^k}\right)\right] =$$

$$= a + abt\lim_{n\to\infty}\left(1 + \sum_{k=1}^{n-2}\frac{b^k}{2^k}\right) = a + abt\frac{1}{1-\frac{b}{2}} = a + \frac{2ab}{2-b}t. \tag{44}$$

For avoiding triviality, the parameter $a \in \mathbb{R}\setminus\{0\}$. The theorem has been proved.

**Solving the integral equation (9)**

**Theorem 4.** The mapping in (9) has the unique fixed point $x^*(t) \in \boldsymbol{X}_{[0;\,1]}$, that is the solution

$$x^*(t) = \frac{2a}{2-b}t \tag{45}$$

of this equation by $a \in \mathbb{R} \setminus \{0\}$ and $|b| < 2$.

**Proof.** Note that the right member in (9) may be factored out with $t$. Applying the solution procedure of the integral equation (6), there are statements

$$x_1(t) = at + b\int_0^1 t x_0(s)ds = t\left(a + b\int_0^1 x_0(s)ds\right) = t\left(a + b\int_0^1 0ds\right) = at\,, \tag{46}$$

$$x_2(t) = t\left(a + b\int_0^1 x_1(s)ds\right) = t\left(a + b\int_0^1 asds\right) = t\left(a + ab\left.\frac{s^2}{2}\right|_0^1\right) = t\left(a + \frac{ab}{2}\right) = at\left(1 + \frac{b}{2}\right), \tag{47}$$

$$x_3(t) = t\left(a + b\int_0^1 x_2(s)ds\right) = t\left(a + b\int_0^1 s\left(a + \frac{ab}{2}\right)ds\right) = t\left(a + ab\left.\frac{s^2}{2}\right|_0^1 + ab^2\left.\frac{s^2}{4}\right|_0^1\right) =$$

$$= t\left(a + \frac{ab}{2} + \frac{ab^2}{4}\right) = at\left(1 + \frac{b}{2} + \frac{b^2}{4}\right), \tag{48}$$

$$x_4(t) = t\left(a + b\int_0^1 x_3(s)ds\right) =$$

$$= t\left(a + b\int_0^1 as\left(1 + \frac{b}{2} + \frac{b^2}{4}\right)ds\right) = t\left(a + ab\left.\frac{s^2}{2}\right|_0^1 + ab^2\left.\frac{s^2}{4}\right|_0^1 + ab^3\left.\frac{s^2}{8}\right|_0^1\right) =$$

$$= t\left(a + \frac{ab}{2} + \frac{ab^2}{4} + \frac{ab^3}{8}\right) = at\left(1 + \frac{b}{2} + \frac{b^2}{4} + \frac{b^3}{8}\right). \tag{49}$$

So, for $n \in \mathbb{N} \setminus \{1\}$ the $n$-th approximation

$$x_n(t) = t\left(a + b\int_0^1 x_{n-1}(s)ds\right) = t\left(a + b\int_0^1 as\left(1 + \sum_{k=1}^{n-2}\frac{b^k}{2^k}\right)ds\right) = at\left(1 + \left[b + b\sum_{k=1}^{n-2}\frac{b^k}{2^k}\right]\left.\frac{s^2}{2}\right|_0^1\right) =$$

$$= at\left(1 + \frac{b}{2} + \frac{b}{2}\sum_{k=1}^{n-2}\frac{b^k}{2^k}\right) = at\left(1 + \frac{b}{2} + \sum_{k=1}^{n-2}\frac{b^{k+1}}{2^{k+1}}\right) = at\left(1 + \sum_{l=1}^{n-1}\frac{b^l}{2^l}\right), \tag{50}$$

giving the geometrical progression in the last member sum. This progression is summable by $|b| < 2$ and then

$$x^*(t) = \lim_{n\to\infty} x_n(t) = \lim_{n\to\infty}\left[at\left(1 + \sum_{l=1}^{n-1}\frac{b^l}{2^l}\right)\right] = at\lim_{n\to\infty}\left(1 + \sum_{l=1}^{n-1}\frac{b^l}{2^l}\right) = \frac{at}{1 - \frac{b}{2}} = \frac{2a}{2-b}t\,. \tag{51}$$

For avoiding triviality, the parameter $a \in \mathbb{R} \setminus \{0\}$. The theorem has been proved.

### Solving the integral equation (10)

**Theorem 5.** The mapping in (10) has the unique fixed point $x^*(t) \in \boldsymbol{X}_{[0;\,1]}$, that is the solution

$$x^*(t) = \frac{2a}{2-b} \tag{52}$$

of this equation by $a \in \mathbb{R} \setminus \{0\}$ and $|b| < 2$.

**Proof.** Not restricted on applying the recurrence (4), take the zeroth approximation (24) and get

$$x_1(t) = a + b\int_0^1 s x_0(s)\,ds = a + b\int_0^1 s \cdot 0 ds = a\,, \tag{53}$$

$$x_2(t) = a + b\int_0^1 s x_1(s)\,ds = a + b\int_0^1 sads = a + ab\left.\frac{s^2}{2}\right|_0^1 = a + \frac{ab}{2}\,, \tag{54}$$

$$x_3(t) = a + b\int_0^1 s x_2(s)\,ds = a + b\int_0^1 s\left(a + \frac{ab}{2}\right)ds = a + ab\left.\frac{s^2}{2}\right|_0^1 + ab^2\left.\frac{s^2}{4}\right|_0^1 = a + \frac{ab}{2} + \frac{ab^2}{4}\,, \tag{55}$$

$$x_4(t) = a + b\int_0^1 s x_3(s)\,ds = a + b\int_0^1 s\left(a + \frac{ab}{2} + \frac{ab^2}{4}\right)ds =$$

$$= a + ab\left.\frac{s^2}{2}\right|_0^1 + ab^2\left.\frac{s^2}{4}\right|_0^1 + ab^3\left.\frac{s^2}{8}\right|_0^1 = a + \frac{ab}{2} + \frac{ab^2}{4} + \frac{ab^3}{8}\,. \tag{56}$$

So, for $n \in \mathbb{N}\setminus\{1\}$ the $n$-th approximation

$$x_n(t) = a + b\int_0^1 s x_{n-1}(s)\,ds = a + b\int_0^1 s\left(a + \sum_{k=1}^{n-2}\frac{ab^k}{2^k}\right)ds =$$

$$= a + \frac{ab}{2} + ab\sum_{k=1}^{n-2}\frac{b^k}{2^k}\cdot\left.\frac{s^2}{2}\right|_0^1 = a + \frac{ab}{2} + ab\sum_{k=1}^{n-2}\frac{b^k}{2^{k+1}} =$$

$$= a + \frac{ab}{2} + a\sum_{k=1}^{n-2}\frac{b^{k+1}}{2^{k+1}} = a + a\sum_{l=1}^{n-1}\frac{b^l}{2^l} = a\left(1 + \sum_{l=1}^{n-1}\frac{b^l}{2^l}\right), \tag{57}$$

giving the geometrical progression in the last member sum. This progression is summable by $|b| < 2$ and then

$$x^*(t) = \lim_{n\to\infty} x_n(t) = \lim_{n\to\infty}\left[a\left(1 + \sum_{l=1}^{n-1}\frac{b^l}{2^l}\right)\right] = a\lim_{n\to\infty}\left(1 + \sum_{l=1}^{n-1}\frac{b^l}{2^l}\right) = a\frac{1}{1 - \frac{b}{2}} = \frac{2a}{2-b}\,, \tag{58}$$

appeared to be just the factor of $t$ in the integral equation (9) solution (45). For avoiding triviality, the parameter $a \in \mathbb{R}\setminus\{0\}$. The theorem has been proved.

**Solving the integral equation (11)**

**Theorem 6.** The mapping in (11) has the unique fixed point $x^*(t) \in \boldsymbol{X}_{[0;\,1]}$, that is the solution

$$x^*(t) = at + \frac{2ab}{3(2-b)} \tag{59}$$

of this equation by $a \in \mathbb{R}\setminus\{0\}$ and $|b| < 2$.

**Proof.** Not restricted on applying the recurrence (4), take the zeroth approximation (24) and get

$$x_1(t) = at + b\int_0^1 s x_0(s)\,ds = at + b\int_0^1 s \cdot 0 ds = at\,, \tag{60}$$

$$x_2(t) = at + b\int_0^1 s x_1(s)\,ds = at + b\int_0^1 sasds = at + ab\left.\frac{s^3}{3}\right|_0^1 = at + \frac{ab}{3}\,, \tag{61}$$

$$x_3(t) = at + b\int_0^1 s x_2(s)\,ds = at + b\int_0^1 s\left(as + \frac{ab}{3}\right)ds = at + ab\left.\frac{s^3}{3}\right|_0^1 + ab^2\left.\frac{s^2}{6}\right|_0^1 = at + \frac{ab}{3} + \frac{ab^2}{6}\,, \tag{62}$$

$$x_4(t) = at + b\int_0^1 s x_3(s)\,ds = at + b\int_0^1 s\left(as + \frac{ab}{3} + \frac{ab^2}{6}\right)ds =$$

$$= at + ab\frac{s^3}{3}\bigg|_0^1 + ab^2\frac{s^2}{6}\bigg|_0^1 + ab^3\frac{s^2}{12}\bigg|_0^1 = at + \frac{ab}{3} + \frac{ab^2}{6} + \frac{ab^3}{12}\,. \tag{63}$$

So, for $n \in \mathbb{N}\setminus\{1\}$ the $n$-th approximation

$$x_n(t) = at + b\int_0^1 s x_{n-1}(s)\,ds = at + b\int_0^1 s\left(as + \sum_{k=1}^{n-2}\frac{ab^k}{3\cdot 2^{k-1}}\right)ds = at + \frac{ab}{3} + ab\sum_{k=1}^{n-2}\frac{b^k}{3\cdot 2^{k-1}}\cdot\frac{s^2}{2}\bigg|_0^1 =$$

$$= at + \frac{ab}{3} + ab\sum_{k=1}^{n-2}\frac{b^k}{3\cdot 2^k} = at + \frac{ab}{3}\left(1 + \sum_{k=1}^{n-2}\frac{b^k}{2^k}\right), \tag{64}$$

giving the geometrical progression in the last member sum. This progression is summable by $|b| < 2$ and then

$$x^*(t) = \lim_{n\to\infty} x_n(t) = \lim_{n\to\infty}\left[at + \frac{ab}{3}\left(1 + \sum_{k=1}^{n-2}\frac{b^k}{2^k}\right)\right] = at + \frac{ab}{3}\lim_{n\to\infty}\left(1 + \sum_{k=1}^{n-2}\frac{b^k}{2^k}\right) = at + \frac{ab}{3}\cdot\frac{1}{1-\frac{b}{2}} = at + \frac{2ab}{3(2-b)}\,. \tag{65}$$

For avoiding triviality, the parameter $a \in \mathbb{R}\setminus\{0\}$. The theorem has been proved.

**Solving the integral equation (12)**

**Theorem 7.** The mapping in (12) has the unique fixed point $x^*(t) \in \boldsymbol{X}_{[0;\,1]}$, that is the solution

$$x^*(t) = a + \frac{3ab}{2(3-b)}t \tag{66}$$

of this equation by $a \in \mathbb{R}\setminus\{0\}$ and $|b| < 3$.

**Proof.** Not restricted on applying the recurrence (4), take the zeroth approximation (24) and get

$$x_1(t) = a + b\int_0^1 ts x_0(s)\,ds = a + b\int_0^1 ts\cdot 0\,ds = a\,, \tag{67}$$

$$x_2(t) = a + b\int_0^1 ts x_1(s)\,ds = a + b\int_0^1 tsa\,ds = a + abt\frac{s^2}{2}\bigg|_0^1 = a + \frac{ab}{2}t\,, \tag{68}$$

$$x_3(t) = a + b\int_0^1 ts x_2(s)\,ds = a + b\int_0^1 ts\left(a + \frac{ab}{2}s\right)ds = a + abt\frac{s^2}{2}\bigg|_0^1 + ab^2t\frac{s^3}{6}\bigg|_0^1 = a + \frac{ab}{2}t + \frac{ab^2}{6}t\,, \tag{69}$$

$$x_4(t) = a + b\int_0^1 ts x_3(s)\,ds = a + b\int_0^1 ts\left(a + \frac{ab}{2}s + \frac{ab^2}{6}s\right)ds =$$

$$= a + abt\frac{s^2}{2}\bigg|_0^1 + ab^2t\frac{s^3}{6}\bigg|_0^1 + ab^3t\frac{s^3}{18}\bigg|_0^1 = a + \frac{ab}{2}t + \frac{ab^2}{6}t + \frac{ab^3}{18}t\,. \tag{70}$$

So, for $n \in \mathbb{N}\setminus\{1\}$ the $n$-th approximation

$$x_n(t) = a + b\int_0^1 ts x_{n-1}(s)\,ds = a + b\int_0^1 ts\left(a + \frac{ab}{2}s\sum_{k=0}^{n-3}\frac{b^k}{3^k}\right)ds = a + abt\cdot\frac{s^2}{2}\bigg|_0^1 + ab^2t\sum_{k=0}^{n-3}\frac{b^k}{3^k}\cdot\frac{s^3}{6}\bigg|_0^1 =$$

$$= a + \frac{ab}{2}t + \frac{ab}{2}t\sum_{k=1}^{n-2}\frac{b^k}{3^k} = a + \frac{ab}{2}t\left(1 + \sum_{k=1}^{n-2}\frac{b^k}{3^k}\right), \tag{71}$$

giving the geometrical progression in the last member sum. This progression is summable by $|b|<3$ and then

$$x^*(t)=\lim_{n\to\infty}x_n(t)=\lim_{n\to\infty}\left[a+\frac{ab}{2}t\left(1+\sum_{k=1}^{n-2}\frac{b^k}{3^k}\right)\right]=a+\frac{ab}{2}t\lim_{n\to\infty}\left(1+\sum_{k=1}^{n-2}\frac{b^k}{3^k}\right)=a+\frac{ab}{2}t\cdot\frac{1}{1-\frac{b}{3}}=a+\frac{3ab}{2(3-b)}t\,. \qquad (72)$$

For avoiding triviality, the parameter $a\in\mathbb{R}\setminus\{0\}$ . The theorem has been proved.

**Solving the integral equation (13)**

**Theorem 8.** The mapping in (13) has the unique fixed point $x^*(t)\in\boldsymbol{X}_{[0;1]}$, that is the solution

$$x^*(t)=\frac{3a}{3-b}t \qquad (73)$$

of this equation by $a\in\mathbb{R}\setminus\{0\}$ and $|b|<3$ .

**Proof.** Not restricted on applying the recurrence (4), take the zeroth approximation (24) and get

$$x_1(t)=at+b\int_0^1 tsx_0(s)\,ds=at+b\int_0^1 ts\cdot 0ds=at\,, \qquad (74)$$

$$x_2(t)=at+b\int_0^1 tsx_1(s)\,ds=at+b\int_0^1 tsasds=at+abt\left.\frac{s^3}{3}\right|_0^1=at+\frac{ab}{3}t\,, \qquad (75)$$

$$x_3(t)=at+b\int_0^1 tsx_2(s)\,ds=at+b\int_0^1 ts\left(as+\frac{ab}{3}s\right)ds=at+abt\left.\frac{s^3}{3}\right|_0^1+ab^2t\left.\frac{s^3}{9}\right|_0^1=at+\frac{ab}{3}t+\frac{ab^2}{9}t\,, \qquad (76)$$

$$x_4(t)=at+b\int_0^1 tsx_3(s)\,ds=at+b\int_0^1 ts\left(as+\frac{ab}{3}s+\frac{ab^2}{9}s\right)ds=$$
$$=at+abt\left.\frac{s^3}{3}\right|_0^1+ab^2t\left.\frac{s^3}{9}\right|_0^1+ab^3t\left.\frac{s^3}{27}\right|_0^1=at+\frac{ab}{3}t+\frac{ab^2}{9}t+\frac{ab^3}{27}t\,. \qquad (77)$$

So, for $n\in\mathbb{N}\setminus\{1\}$ the $n$-th approximation

$$x_n(t)=at+b\int_0^1 tsx_{n-1}(s)\,ds=at+b\int_0^1 ts\left(as+\frac{ab}{3}s\sum_{k=0}^{n-3}\frac{b^k}{3^k}\right)ds=at+abt\left.\frac{s^3}{3}\right|_0^1+ab^2t\sum_{k=0}^{n-3}\frac{b^k}{3^k}\cdot\left.\frac{s^3}{9}\right|_0^1=$$
$$=at+\frac{ab}{3}t+\frac{ab^2}{9}t\sum_{k=0}^{n-3}\frac{b^k}{3^k}=at\left(1+\frac{b}{3}+\frac{b^2}{9}\sum_{k=0}^{n-3}\frac{b^k}{3^k}\right)=at\left(1+\frac{b}{3}\left[1+\frac{b}{3}\sum_{k=0}^{n-3}\frac{b^k}{3^k}\right]\right)=at\left(1+\frac{b}{3}\left[1+\sum_{l=1}^{n-2}\frac{b^l}{3^l}\right]\right), \qquad (78)$$

giving the geometrical progression in the last member sum. This progression is summable by $|b|<3$ and then

$$x^*(t)=\lim_{n\to\infty}x_n(t)=\lim_{n\to\infty}\left[at\left(1+\frac{b}{3}\left[1+\sum_{l=1}^{n-2}\frac{b^l}{3^l}\right]\right)\right]=at\left(1+\frac{b}{3}\lim_{n\to\infty}\left[1+\sum_{l=1}^{n-2}\frac{b^l}{3^l}\right]\right)=$$
$$=at\left(1+\frac{b}{3}\cdot\frac{1}{1-\frac{b}{3}}\right)=at\left(1+\frac{b}{3}\cdot\frac{3}{3-b}\right)=at\left(1+\frac{b}{3-b}\right)=\frac{3a}{3-b}t\,. \qquad (79)$$

For avoiding triviality, the parameter $a\in\mathbb{R}\setminus\{0\}$ . The theorem has been proved.

**Solving the integral equation (14), generalizing the equations (6) and (8)**

**Theorem 9.** The mapping in (14) has the unique fixed point $x^*(t) \in \boldsymbol{X}_{[0;1]}$, that is the solution

$$x^*(t) = a + \frac{ab(q+1)}{q+1-b} t^q \tag{80}$$

of this equation by $a \in \mathbb{R} \setminus \{0\}$ and $|b| < q+1$.

**Proof.** Not restricted on applying the recurrence (4), take the zeroth approximation (24) and get

$$x_1(t) = a + b\int_0^1 t^q x_0(s)\,ds = a + b\int_0^1 t^q \cdot 0 ds = a \,, \tag{81}$$

$$x_2(t) = a + b\int_0^1 t^q x_1(s)\,ds = a + b\int_0^1 t^q a ds = a + abt^q \,, \tag{82}$$

$$x_3(t) = a + b\int_0^1 t^q x_2(s)\,ds = a + b\int_0^1 t^q\left(a + abs^q\right)ds = a + abt^q + ab^2 t^q \left.\frac{s^{q+1}}{q+1}\right|_0^1 = a + abt^q + \frac{ab^2}{q+1} t^q \,, \tag{83}$$

$$x_4(t) = a + b\int_0^1 t^q x_3(s)\,ds = a + b\int_0^1 t^q \left(a + abs^q + \frac{ab^2}{q+1} s^q\right)ds =$$

$$= a + abt^q + ab^2 t^q \left.\frac{s^{q+1}}{q+1}\right|_0^1 + ab^3 t^q \left.\frac{s^{q+1}}{(q+1)^2}\right|_0^1 = a + abt^q + \frac{ab^2}{q+1} t^q + \frac{ab^3}{(q+1)^2} t^q \,. \tag{84}$$

So, for $n \in \mathbb{N} \setminus \{1\}$ the $n$-th approximation

$$x_n(t) = a + b\int_0^1 t^q x_{n-1}(s)\,ds = a + b\int_0^1 t^q \left(a + \sum_{k=1}^{n-2} \frac{ab^k s^q}{(q+1)^{k-1}}\right)ds = a + abt^q + abt^q \sum_{k=1}^{n-2} \frac{b^k}{(q+1)^{k-1}} \cdot \left.\frac{s^{q+1}}{q+1}\right|_0^1 =$$

$$= a + abt^q + abt^q \sum_{k=1}^{n-2} \frac{b^k}{(q+1)^k} = a + abt^q \left(1 + \sum_{k=1}^{n-2} \frac{b^k}{(q+1)^k}\right), \tag{85}$$

giving the geometrical progression in the last member sum. This progression is summable by $|b| < q+1$ and then

$$x^*(t) = \lim_{n\to\infty} x_n(t) = \lim_{n\to\infty}\left[a + abt^q \left(1 + \sum_{k=1}^{n-2} \frac{b^k}{(q+1)^k}\right)\right] = a + abt^q \lim_{n\to\infty}\left(1 + \sum_{k=1}^{n-2} \frac{b^k}{(q+1)^k}\right) =$$

$$= a + abt^q \frac{1}{1 - \frac{b}{q+1}} = a + \frac{ab(q+1)}{q+1-b} t^q \,. \tag{86}$$

For avoiding triviality, the parameter $a \in \mathbb{R} \setminus \{0\}$. The theorem, generalizing the Theorem 1 and Theorem 3, has been proved.

**Solving the integral equation (15), generalizing the equations (7) and (9)**

**Theorem 10.** The mapping in (15) has the unique fixed point $x^*(t) \in \boldsymbol{X}_{[0;1]}$, that is the solution

$$x^*(t) = at + \frac{ab(q+1)}{2(q+1-b)} t^q \tag{87}$$

of this equation by $a \in \mathbb{R} \setminus \{0\}$ and $|b| < q+1$.

**Proof.** Not restricted on applying the recurrence (4), take the zeroth approximation (24) and get

$$x_1(t) = at + b\int_0^1 t^q x_0(s)\,ds = at + b\int_0^1 t^q \cdot 0\,ds = at\,, \tag{88}$$

$$x_2(t) = at + b\int_0^1 t^q x_1(s)\,ds = at + b\int_0^1 t^q as\,ds = at + abt^q \left.\frac{s^2}{2}\right|_0^1 = at + \frac{ab}{2}t^q\,, \tag{89}$$

$$x_3(t) = at + b\int_0^1 t^q x_2(s)\,ds = at + b\int_0^1 t^q \left( as + \frac{ab}{2}s^q \right) ds = at + abt^q \left.\frac{s^2}{2}\right|_0^1 + ab^2 t^q \left.\frac{s^{q+1}}{2(q+1)}\right|_0^1 =$$

$$= at + \frac{ab}{2}t^q + \frac{ab^2}{2(q+1)}t^q\,, \tag{90}$$

$$x_4(t) = at + b\int_0^1 t^q x_3(s)\,ds = at + b\int_0^1 t^q \left( as + \frac{ab}{2}s^q + \frac{ab^2}{2(q+1)}s^q \right) ds =$$

$$= at + abt^q \left.\frac{s^2}{2}\right|_0^1 + ab^2 t^q \left.\frac{s^{q+1}}{2(q+1)}\right|_0^1 + ab^3 t^q \left.\frac{s^{q+1}}{2(q+1)^2}\right|_0^1 = at + \frac{ab}{2}t^q + \frac{ab^2}{2(q+1)}t^q + \frac{ab^3}{2(q+1)^2}t^q\,. \tag{91}$$

So, for $n \in \mathbb{N} \setminus \{1\}$ the $n$-th approximation

$$x_n(t) = at + b\int_0^1 t^q x_{n-1}(s)\,ds = at + b\int_0^1 t^q \left( as + \frac{ab}{2}\sum_{k=0}^{n-3} \frac{b^k}{(q+1)^k} s^q \right) ds =$$

$$= at + abt^q \left.\frac{s^2}{2}\right|_0^1 + ab^2 t^q \sum_{k=0}^{n-3} \frac{b^k}{(q+1)^k} \cdot \left.\frac{s^{q+1}}{2(q+1)}\right|_0^1 = at + \frac{ab}{2}t^q + \frac{ab}{2}t^q \sum_{k=0}^{n-3} \frac{b^{k+1}}{(q+1)^{k+1}} =$$

$$= at + \frac{ab}{2}t^q + \frac{ab}{2}t^q \sum_{l=1}^{n-2} \frac{b^l}{(q+1)^l} = at + \frac{ab}{2}t^q \left( 1 + \sum_{l=1}^{n-2} \frac{b^l}{(q+1)^l} \right), \tag{92}$$

giving the geometrical progression in the last member sum. This progression is summable by $|b| < q+1$ and then

$$x^*(t) = \lim_{n\to\infty} x_n(t) = \lim_{n\to\infty} \left[ at + \frac{ab}{2}t^q \left( 1 + \sum_{l=1}^{n-2} \frac{b^l}{(q+1)^l} \right) \right] = at + \frac{ab}{2}t^q \lim_{n\to\infty} \left( 1 + \sum_{l=1}^{n-2} \frac{b^l}{(q+1)^l} \right) =$$

$$= at + \frac{ab}{2}t^q \cdot \frac{1}{1 - \dfrac{b}{q+1}} = at + \frac{ab(q+1)}{2(q+1-b)}t^q\,. \tag{93}$$

For avoiding triviality, the parameter $a \in \mathbb{R} \setminus \{0\}$. The theorem, generalizing the Theorem 2 and Theorem 4, has been proved.

**Solving the integral equation (16), generalizing the equations (6) and (10)**

**Theorem 11.** The mapping in (16) has the unique fixed point $x^*(t) \in \boldsymbol{X}_{[0;\,1]}$, that is the solution

$$x^*(t) = \frac{a(m+1)}{m+1-b} \tag{94}$$

of this equation by $a \in \mathbb{R} \setminus \{0\}$ and $|b| < m+1$.

**Proof.** Not restricted on applying the recurrence (4), take the zeroth approximation (24) and get

$$x_1(t) = a + b\int_0^1 s^m x_0(s)\,ds = a + b\int_0^1 s^m \cdot 0\,ds = a\,, \tag{95}$$

$$x_2(t)=a+b\int_0^1 s^m x_1(s)\,ds=a+b\int_0^1 s^m a\,ds=a+ab\left.\frac{s^{m+1}}{m+1}\right|_0^1=a+\frac{ab}{m+1}\,, \tag{96}$$

$$x_3(t)=a+b\int_0^1 s^m x_2(s)\,ds=a+b\int_0^1 s^m\left(a+\frac{ab}{m+1}\right)ds=$$
$$=a+ab\left.\frac{s^{m+1}}{m+1}\right|_0^1+ab^2\left.\frac{s^{m+1}}{(m+1)^2}\right|_0^1=a+\frac{ab}{m+1}+\frac{ab^2}{(m+1)^2}\,, \tag{97}$$

$$x_4(t)=a+b\int_0^1 s^m x_3(s)\,ds=a+b\int_0^1 s^m\left(a+\frac{ab}{m+1}+\frac{ab^2}{(m+1)^2}\right)ds=$$
$$=a+ab\left.\frac{s^{m+1}}{m+1}\right|_0^1+ab^2\left.\frac{s^{m+1}}{(m+1)^2}\right|_0^1+ab^3\left.\frac{s^{m+1}}{(m+1)^3}\right|_0^1=a+\frac{ab}{m+1}+\frac{ab^2}{(m+1)^2}+\frac{ab^3}{(m+1)^3}\,. \tag{98}$$

So, for $n\in\mathbb{N}\setminus\{1\}$ the $n$-th approximation

$$x_n(t)=a+b\int_0^1 s^m x_{n-1}(s)\,ds=$$
$$=a+b\int_0^1 s^m\left(a+\sum_{k=1}^{n-2}\frac{ab^k}{(m+1)^k}\right)ds=a+ab\left.\frac{s^{m+1}}{m+1}\right|_0^1+ab\sum_{k=1}^{n-2}\frac{b^k}{(m+1)^k}\cdot\left.\frac{s^{m+1}}{m+1}\right|_0^1=$$
$$=a+\frac{ab}{m+1}+a\sum_{k=1}^{n-2}\frac{b^{k+1}}{(m+1)^{k+1}}=a+a\sum_{l=1}^{n-1}\frac{b^l}{(m+1)^l}=a\left(1+\sum_{l=1}^{n-1}\frac{b^l}{(m+1)^l}\right), \tag{99}$$

giving the geometrical progression in the last member sum. This progression is summable by $|b|<m+1$ and then

$$x^*(t)=\lim_{n\to\infty}x_n(t)=\lim_{n\to\infty}\left[a\left(1+\sum_{l=1}^{n-1}\frac{b^l}{(m+1)^l}\right)\right]=a\lim_{n\to\infty}\left(1+\sum_{l=1}^{n-1}\frac{b^l}{(m+1)^l}\right)=a\frac{1}{1-\frac{b}{m+1}}=\frac{a(m+1)}{m+1-b}\,. \tag{100}$$

For avoiding triviality, the parameter $a\in\mathbb{R}\setminus\{0\}$. The theorem, generalizing the Theorem 1 and Theorem 5, has been proved.

**Solving the integral equation (17), generalizing the equations (7) and (11)**

**Theorem 12.** The mapping in (17) has the unique fixed point $x^*(t)\in\boldsymbol{X}_{[0;\,1]}$, that is the solution

$$x^*(t)=at+\frac{ab(m+1)}{(m+2)(m+1-b)} \tag{101}$$

of this equation by $a\in\mathbb{R}\setminus\{0\}$ and $|b|<m+1$.

**Proof.** Not restricted on applying the recurrence (4), take the zeroth approximation (24) and get

$$x_1(t)=at+b\int_0^1 s^m x_0(s)\,ds=at+b\int_0^1 s^m\cdot 0\,ds=at\,, \tag{102}$$

$$x_2(t)=at+b\int_0^1 s^m x_1(s)\,ds=at+b\int_0^1 s^m as\,ds=at+ab\left.\frac{s^{m+2}}{m+2}\right|_0^1=at+\frac{ab}{m+2}\,, \tag{103}$$

$$x_3(t)=at+b\int_0^1 s^m x_2(s)\,ds=$$

$$= at + b\int_0^1 s^m\left(as + \frac{ab}{m+2}\right)ds = at + ab\left.\frac{s^{m+2}}{m+2}\right|_0^1 + ab^2\left.\frac{s^{m+1}}{(m+1)(m+2)}\right|_0^1 =$$

$$= at + \frac{ab}{m+2} + \frac{ab^2}{(m+1)(m+2)}, \tag{104}$$

$$x_4(t) = at + b\int_0^1 s^m x_3(s)ds = at + b\int_0^1 s^m\left(as + \frac{ab}{m+2} + \frac{ab^2}{(m+1)(m+2)}\right)ds =$$

$$= at + ab\left.\frac{s^{m+2}}{m+2}\right|_0^1 + ab^2\left.\frac{s^{m+1}}{(m+1)(m+2)}\right|_0^1 + ab^3\left.\frac{s^{m+1}}{(m+1)^2(m+2)}\right|_0^1 =$$

$$= at + \frac{ab}{m+2} + \frac{ab^2}{(m+1)(m+2)} + \frac{ab^3}{(m+1)^2(m+2)}. \tag{105}$$

So, for $n \in \mathbb{N}\setminus\{1\}$ the $n$-th approximation

$$x_n(t) = at + b\int_0^1 s^m x_{n-1}(s)ds =$$

$$= at + b\int_0^1 s^m\left[as + a\sum_{k=1}^{n-2}\frac{b^k}{(m+2)(m+1)^{k-1}}\right]ds = at + \frac{ab}{m+2} + \frac{ab}{m+2}\sum_{k=1}^{n-2}\frac{b^k}{(m+1)^{k-1}}\cdot\left.\frac{s^{m+1}}{m+1}\right|_0^1 =$$

$$= at + \frac{ab}{m+2} + \frac{ab}{m+2}\sum_{k=1}^{n-2}\frac{b^k}{(m+1)^k} = at + \frac{ab}{m+2}\left(1 + \sum_{k=1}^{n-2}\frac{b^k}{(m+1)^k}\right), \tag{106}$$

giving the geometrical progression in the last member sum. This progression is summable by $|b| < m+1$ and then

$$x^*(t) = \lim_{n\to\infty} x_n(t) = \lim_{n\to\infty}\left[at + \frac{ab}{m+2}\left(1 + \sum_{k=1}^{n-2}\frac{b^k}{(m+1)^k}\right)\right] =$$

$$= at + \frac{ab}{m+2}\lim_{n\to\infty}\left(1 + \sum_{k=1}^{n-2}\frac{b^k}{(m+1)^k}\right) = at + \frac{ab}{m+2}\cdot\frac{1}{1 - \frac{b}{m+1}} = at + \frac{ab(m+1)}{(m+2)(m+1-b)}. \tag{107}$$

For avoiding triviality, the parameter $a \in \mathbb{R}\setminus\{0\}$. The theorem, generalizing the Theorem 2 and Theorem 6, has been proved.

**Solving the integral equation (18), generalizing the equations (8) and (12)**

**Theorem 13.** The mapping in (18) has the unique fixed point $x^*(t) \in \boldsymbol{X}_{[0;\,1]}$, that is the solution

$$x^*(t) = a + \frac{ab(m+2)}{(m+1)(m+2-b)}t \tag{108}$$

of this equation by $a \in \mathbb{R}\setminus\{0\}$ and $|b| < m+2$.

**Proof.** Not restricted on applying the recurrence (4), take the zeroth approximation (24) and get

$$x_1(t) = a + b\int_0^1 ts^m x_0(s)ds = a + b\int_0^1 ts^m\cdot 0ds = a, \tag{109}$$

$$x_2(t) = a + b\int_0^1 ts^m x_1(s)ds = a + b\int_0^1 ts^m a ds = a + abt\left.\frac{s^{m+1}}{m+1}\right|_0^1 = a + \frac{ab}{m+1}t, \tag{110}$$

$$x_3(t)=a+b\int_0^1 ts^m x_2(s)ds=a+b\int_0^1 ts^m\left(a+\frac{ab}{m+1}s\right)ds=$$

$$=a+abt\left.\frac{s^{m+1}}{m+1}\right|_0^1+ab^2t\left.\frac{s^{m+2}}{(m+1)(m+2)}\right|_0^1=a+\frac{ab}{m+1}t+\frac{ab^2}{(m+1)(m+2)}t\ , \tag{111}$$

$$x_4(t)=a+b\int_0^1 ts^m x_3(s)ds=a+b\int_0^1 ts^m\left(a+\frac{ab}{m+1}s+\frac{ab^2}{(m+1)(m+2)}s\right)ds=$$

$$=a+abt\left.\frac{s^{m+1}}{m+1}\right|_0^1+ab^2t\left.\frac{s^{m+2}}{(m+1)(m+2)}\right|_0^1+ab^3t\left.\frac{s^{m+2}}{(m+1)(m+2)^2}\right|_0^1=$$

$$=a+\frac{ab}{m+1}t+\frac{ab^2}{(m+1)(m+2)}t+\frac{ab^3}{(m+1)(m+2)^2}t\ . \tag{112}$$

So, for $n\in\mathbb{N}\setminus\{1\}$ the $n$-th approximation

$$x_n(t)=a+b\int_0^1 ts^m x_{n-1}(s)ds=a+b\int_0^1 ts^m\left(a+\frac{ab}{m+1}s\sum_{k=0}^{n-3}\frac{b^k}{(m+2)^k}\right)ds=$$

$$=a+abt\left.\frac{s^{m+1}}{m+1}\right|_0^1+\frac{ab^2t}{m+1}\sum_{k=0}^{n-3}\frac{b^k}{(m+2)^k}\cdot\left.\frac{s^{m+2}}{m+2}\right|_0^1=$$

$$=a+\frac{ab}{m+1}t+\frac{ab}{m+1}t\sum_{k=1}^{n-2}\frac{b^k}{(m+2)^k}=a+\frac{ab}{m+1}t\left(1+\sum_{k=1}^{n-2}\frac{b^k}{(m+2)^k}\right), \tag{113}$$

giving the geometrical progression in the last member sum. This progression is summable by $|b|<m+2$ and then

$$x^*(t)=\lim_{n\to\infty}x_n(t)=\lim_{n\to\infty}\left[a+\frac{ab}{m+1}t\left(1+\sum_{k=1}^{n-2}\frac{b^k}{(m+2)^k}\right)\right]=a+\frac{ab}{m+1}t\lim_{n\to\infty}\left(1+\sum_{k=1}^{n-2}\frac{b^k}{(m+2)^k}\right)=$$

$$=a+\frac{abt}{m+1}\cdot\frac{1}{1-\frac{b}{m+2}}=a+\frac{ab(m+2)}{(m+1)(m+2-b)}t\ . \tag{114}$$

For avoiding triviality, the parameter $a\in\mathbb{R}\setminus\{0\}$. The theorem, generalizing the Theorem 3 and Theorem 7, has been proved.

**Solving the integral equation (19), generalizing the equations (9) and (13)**

**Theorem 14.** The mapping in (19) has the unique fixed point $x^*(t)\in\boldsymbol{X}_{[0;\,1]}$, that is the solution

$$x^*(t)=\frac{a(m+2)}{m+2-b}t \tag{115}$$

of this equation by $a\in\mathbb{R}\setminus\{0\}$ and $|b|<m+2$.

**Proof.** Not restricted on applying the recurrence (4), take the zeroth approximation (24) and get

$$x_1(t)=at+b\int_0^1 ts^m x_0(s)ds=at+b\int_0^1 ts^m\cdot 0ds=at\ , \tag{116}$$

$$x_2(t)=at+b\int_0^1 ts^m x_1(s)ds=at+b\int_0^1 ts^m asds=at+abt\left.\frac{s^{m+2}}{m+2}\right|_0^1=at+\frac{ab}{m+2}t\ , \tag{117}$$

$$x_3(t) = at + b\int_0^1 ts^m x_2(s)\,ds = at + b\int_0^1 ts^m \left( as + \frac{ab}{m+2}s \right) ds =$$

$$= at + abt\frac{s^{m+2}}{m+2}\Bigg|_0^1 + ab^2 t\frac{s^{m+2}}{(m+2)^2}\Bigg|_0^1 = at + \frac{ab}{m+2}t + \frac{ab^2}{(m+2)^2}t \;, \tag{118}$$

$$x_4(t) = at + b\int_0^1 ts^m x_3(s)\,ds = at + b\int_0^1 ts^m \left( as + \frac{ab}{m+2}s + \frac{ab^2}{(m+2)^2}s \right) ds =$$

$$= at + abt\frac{s^{m+2}}{m+2}\Bigg|_0^1 + ab^2 t\frac{s^{m+2}}{(m+2)^2}\Bigg|_0^1 + ab^3 t\frac{s^{m+2}}{(m+2)^3}\Bigg|_0^1 = at + \frac{ab}{m+2}t + \frac{ab^2}{(m+2)^2}t + \frac{ab^3}{(m+2)^3}t \;. \tag{119}$$

So, for $n \in \mathbb{N} \setminus \{1\}$ the $n$-th approximation

$$x_n(t) = at + b\int_0^1 ts^m x_{n-1}(s)\,ds = at + b\int_0^1 ts^m \left( as + as\sum_{k=1}^{n-2}\frac{b^k}{(m+2)^k} \right) ds =$$

$$= at + abt\frac{s^{m+2}}{m+2}\Bigg|_0^1 + abt\sum_{k=1}^{n-2}\frac{b^k}{(m+2)^k}\cdot\frac{s^{m+2}}{m+2}\Bigg|_0^1 =$$

$$= at + \frac{ab}{m+2}t + \frac{ab}{m+2}t\sum_{k=1}^{n-2}\frac{b^k}{(m+2)^k} = at + \frac{ab}{m+2}t + at\sum_{k=1}^{n-2}\frac{b^{k+1}}{(m+2)^{k+1}} =$$

$$= at\left( 1 + \frac{b}{m+2} + \sum_{k=1}^{n-2}\frac{b^{k+1}}{(m+2)^{k+1}} \right) = at\left( 1 + \sum_{l=1}^{n-1}\frac{b^l}{(m+2)^l} \right), \tag{120}$$

giving the geometrical progression in the last member sum. This progression is summable by $|b| < m+2$ and then

$$x^*(t) = \lim_{n\to\infty} x_n(t) = \lim_{n\to\infty}\left[ at\left( 1 + \sum_{l=1}^{n-1}\frac{b^l}{(m+2)^l} \right) \right] = at\lim_{n\to\infty}\left( 1 + \sum_{l=1}^{n-1}\frac{b^l}{(m+2)^l} \right) =$$

$$= at\frac{1}{1 - \dfrac{b}{m+2}} = at\frac{m+2}{m+2-b} = \frac{a(m+2)}{m+2-b}t \;. \tag{121}$$

For avoiding triviality, the parameter $a \in \mathbb{R} \setminus \{0\}$ . The theorem, generalizing the Theorem 4 and Theorem 8, has been proved.

**Solving the integral equation (20), generalizing the equation (18)**

**Theorem 15.** The mapping in (20) has the unique fixed point $x^*(t) \in \boldsymbol{X}_{[0;\,1]}$, that is the solution

$$x^*(t) = a + \frac{ab(m+q+1)}{(m+1)(m+q+1-b)}t^q \tag{122}$$

of this equation by $a \in \mathbb{R} \setminus \{0\}$ and $|b| < m+q+1$ .

**Proof.** Not restricted on applying the recurrence (4), take the zeroth approximation (24) and get

$$x_1(t) = a + b\int_0^1 t^q s^m x_0(s)\,ds = a + b\int_0^1 t^q s^m \cdot 0\,ds = a \;, \tag{123}$$

$$x_2(t) = a + b\int_0^1 t^q s^m x_1(s)\,ds = a + b\int_0^1 t^q s^m a\,ds = a + abt^q\frac{s^{m+1}}{m+1}\Bigg|_0^1 = a + \frac{ab}{m+1}t^q \;, \tag{124}$$

$$x_3(t)=a+b\int_0^1 t^q s^m x_2(s)\,ds=a+b\int_0^1 t^q s^m\left(a+\frac{ab}{m+1}s^q\right)ds=$$

$$=a+abt^q\left.\frac{s^{m+1}}{m+1}\right|_0^1+ab^2t^q\left.\frac{s^{m+q+1}}{(m+1)(m+q+1)}\right|_0^1=a+\frac{ab}{m+1}t^q+\frac{ab^2}{(m+1)(m+q+1)}t^q\,, \tag{125}$$

$$x_4(t)=a+b\int_0^1 t^q s^m x_3(s)\,ds=a+b\int_0^1 t^q s^m\left(a+\frac{ab}{m+1}s^q+\frac{ab^2}{(m+1)(m+q+1)}s^q\right)ds=$$

$$=a+abt^q\left.\frac{s^{m+1}}{m+1}\right|_0^1+ab^2t^q\left.\frac{s^{m+q+1}}{(m+1)(m+q+1)}\right|_0^1+ab^3t^q\left.\frac{s^{m+q+1}}{(m+1)(m+q+1)^2}\right|_0^1=$$

$$=a+\frac{ab}{m+1}t^q+\frac{ab^2}{(m+1)(m+q+1)}t^q+\frac{ab^3}{(m+1)(m+q+1)^2}t^q\,. \tag{126}$$

So, for $n\in\mathbb{N}\setminus\{1\}$ the $n$-th approximation

$$x_n(t)=a+b\int_0^1 t^q s^m x_{n-1}(s)\,ds=a+b\int_0^1 t^q s^m\left(a+\frac{ab}{m+1}s^q\sum_{k=0}^{n-3}\frac{b^k}{(m+q+1)^k}\right)ds=$$

$$=a+abt^q\left.\frac{s^{m+1}}{m+1}\right|_0^1+\frac{ab^2t^q}{m+1}\sum_{k=0}^{n-3}\frac{b^k}{(m+q+1)^k}\cdot\left.\frac{s^{m+q+1}}{m+q+1}\right|_0^1=$$

$$=a+\frac{ab}{m+1}t^q+\frac{ab}{m+1}t^q\sum_{k=1}^{n-2}\frac{b^k}{(m+q+1)^k}=a+\frac{ab}{m+1}t^q\left(1+\sum_{k=1}^{n-2}\frac{b^k}{(m+q+1)^k}\right), \tag{127}$$

giving the geometrical progression in the last member sum. This progression is summable by $|b|<m+q+1$ and then

$$x^*(t)=\lim_{n\to\infty}x_n(t)=\lim_{n\to\infty}\left[a+\frac{ab}{m+1}t^q\left(1+\sum_{k=1}^{n-2}\frac{b^k}{(m+q+1)^k}\right)\right]=$$

$$=a+\frac{ab}{m+1}t^q\lim_{n\to\infty}\left(1+\sum_{k=1}^{n-2}\frac{b^k}{(m+q+1)^k}\right)=$$

$$=a+\frac{abt^q}{m+1}\cdot\frac{1}{1-\dfrac{b}{m+q+1}}=a+\frac{ab(m+q+1)}{(m+1)(m+q+1-b)}t^q\,. \tag{128}$$

For avoiding triviality, the parameter $a\in\mathbb{R}\setminus\{0\}$ . The theorem, generalizing the Theorem 13, has been proved.

### Solving the integral equation (21), generalizing the equation (19)

**Theorem 16.** The mapping in (21) has the unique fixed point $x^*(t)\in\boldsymbol{X}_{[0;\,1]}$, that is the solution

$$x^*(t)=at+\frac{ab(m+q+1)}{(m+2)(m+q+1-b)}t^q \tag{129}$$

of this equation by $a\in\mathbb{R}\setminus\{0\}$ and $|b|<m+q+1$ .

**Proof.** Not restricted on applying the recurrence (4), take the zeroth approximation (24) and get

$$x_1(t)=at+b\int_0^1 t^q s^m x_0(s)\,ds=at+b\int_0^1 t^q s^m\cdot 0\,ds=at\,, \tag{130}$$

$$x_2(t)=at+b\int_0^1 t^q s^m x_1(s)\,ds=at+b\int_0^1 t^q s^m as\,ds=at+abt^q\left.\frac{s^{m+2}}{m+2}\right|_0^1=at+\frac{ab}{m+2}t^q\,, \tag{131}$$

$$x_3(t)=at+b\int_0^1 t^q s^m x_2(s)\,ds=at+b\int_0^1 t^q s^m\left(as+\frac{ab}{m+2}s^q\right)ds=$$

$$=at+abt^q\left.\frac{s^{m+2}}{m+2}\right|_0^1+ab^2t^q\left.\frac{s^{m+q+1}}{(m+2)(m+q+1)}\right|_0^1=at+\frac{ab}{m+2}t^q+\frac{ab^2}{(m+2)(m+q+1)}t^q\,, \tag{132}$$

$$x_4(t)=at+b\int_0^1 t^q s^m x_3(s)\,ds=at+b\int_0^1 t^q s^m\left[as+\frac{ab}{m+2}s^q+\frac{ab^2}{(m+2)(m+q+1)}s^q\right]ds=$$

$$=at+abt^q\left.\frac{s^{m+2}}{m+2}\right|_0^1+ab^2t^q\left.\frac{s^{m+q+1}}{(m+2)(m+q+1)}\right|_0^1+ab^3t^q\left.\frac{s^{m+q+1}}{(m+2)(m+q+1)^2}\right|_0^1=$$

$$=at+\frac{ab}{m+2}t^q+\frac{ab^2}{(m+2)(m+q+1)}t^q+\frac{ab^3}{(m+2)(m+q+1)^2}t^q\,. \tag{133}$$

So, for $n\in\mathbb{N}\setminus\{1\}$ the $n$-th approximation

$$x_n(t)=at+b\int_0^1 t^q s^m x_{n-1}(s)\,ds=at+b\int_0^1 t^q s^m\left(as+as^q\sum_{k=1}^{n-2}\frac{b^k}{(m+2)(m+q+1)^{k-1}}\right)ds=$$

$$=at+abt^q\left.\frac{s^{m+2}}{m+2}\right|_0^1+\frac{abt^q}{m+2}\sum_{k=1}^{n-2}\frac{b^k}{(m+q+1)^{k-1}}\cdot\left.\frac{s^{m+q+1}}{m+q+1}\right|_0^1=$$

$$=at+\frac{ab}{m+2}t^q+\frac{ab}{m+2}t^q\sum_{k=1}^{n-2}\frac{b^k}{(m+q+1)^k}=at+\frac{ab}{m+2}t^q\left(1+\sum_{k=1}^{n-2}\frac{b^k}{(m+q+1)^k}\right), \tag{134}$$

giving the geometrical progression in the last member sum. This progression is summable by $|b|<m+q+1$ and then

$$x^*(t)=\lim_{n\to\infty}x_n(t)=\lim_{n\to\infty}\left[at+\frac{ab}{m+2}t^q\left(1+\sum_{k=1}^{n-2}\frac{b^k}{(m+q+1)^k}\right)\right]=$$

$$=at+\frac{ab}{m+2}t^q\lim_{n\to\infty}\left(1+\sum_{k=1}^{n-2}\frac{b^k}{(m+q+1)^k}\right)=$$

$$=at+\frac{ab}{m+2}t^q\cdot\frac{1}{1-\dfrac{b}{m+q+1}}=at+\frac{ab(m+q+1)}{(m+2)(m+q+1-b)}t^q\,. \tag{135}$$

For avoiding triviality, the parameter $a\in\mathbb{R}\setminus\{0\}$. The theorem, generalizing the Theorem 14, has been proved.

**Solving the integral equation (22), generalizing the equation (21) globally in polynomializing $f(t)$ part**

**Theorem 17.** The mapping in (22) has the unique fixed point $x^*(t)\in\boldsymbol{X}_{[0;\,1]}$, that is the solution

$$x^*(t)=\sum_{i=0}^{p}a_i t^i+\frac{b(m+q+1)}{m+q+1-b}t^q\cdot\sum_{i=0}^{p}\frac{a_i}{m+i+1} \tag{136}$$

of this equation by $a_i\in\mathbb{R}\ \ \forall\, i=\overline{0,\,p}$ in $\exists\, i_0\in\left\{\overline{0,\,p}\right\}$ with $a_{i_0}\in\mathbb{R}\setminus\{0\}$ and $|b|<m+q+1$.

**Proof.** Not restricted on applying the recurrence (4), take the zeroth approximation (24) and get

$$x_1(t)=\sum_{i=0}^{p}a_it^i+b\int_0^1 t^qs^mx_0(s)ds=\sum_{i=0}^{p}a_it^i+b\int_0^1 t^qs^m\cdot 0ds=\sum_{i=0}^{p}a_it^i\,, \tag{137}$$

$$x_2(t)=\sum_{i=0}^{p}a_it^i+b\int_0^1 t^qs^mx_1(s)ds=\sum_{i=0}^{p}a_it^i+b\int_0^1 t^qs^m\left(\sum_{i=0}^{p}a_is^i\right)ds=\sum_{i=0}^{p}a_it^i+bt^q\sum_{i=0}^{p}\left(a_i\int_0^1 s^{m+i}ds\right)=$$

$$=\sum_{i=0}^{p}a_it^i+bt^q\sum_{i=0}^{p}\left(a_i\left.\frac{s^{m+i+1}}{m+i+1}\right|_0^1\right)=\sum_{i=0}^{p}a_it^i+bt^q\sum_{i=0}^{p}\frac{a_i}{m+i+1}\,, \tag{138}$$

$$x_3(t)=\sum_{i=0}^{p}a_it^i+b\int_0^1 t^qs^mx_2(s)ds=\sum_{i=0}^{p}a_it^i+b\int_0^1 t^qs^m\left(\sum_{i=0}^{p}a_is^i+bs^q\sum_{i=0}^{p}\frac{a_i}{m+i+1}\right)ds=$$

$$=\sum_{i=0}^{p}a_it^i+bt^q\left[\sum_{i=0}^{p}\left(a_i\int_0^1 s^{m+i}ds\right)+\sum_{i=0}^{p}\left(\frac{a_ib}{m+i+1}\int_0^1 s^{m+q}ds\right)\right]=$$

$$=\sum_{i=0}^{p}a_it^i+bt^q\left[\sum_{i=0}^{p}\left(a_i\left.\frac{s^{m+i+1}}{m+i+1}\right|_0^1\right)+\sum_{i=0}^{p}\left(a_ib\left.\frac{s^{m+q+1}}{(m+i+1)(m+q+1)}\right|_0^1\right)\right]=$$

$$=\sum_{i=0}^{p}a_it^i+bt^q\sum_{i=0}^{p}\frac{a_i}{m+i+1}+b^2t^q\sum_{i=0}^{p}\frac{a_i}{(m+i+1)(m+q+1)}=$$

$$=\sum_{i=0}^{p}a_it^i+bt^q\sum_{i=0}^{p}\frac{a_i}{m+i+1}+\frac{b^2t^q}{m+q+1}\sum_{i=0}^{p}\frac{a_i}{m+i+1}\,, \tag{139}$$

$$x_4(t)=\sum_{i=0}^{p}a_it^i+b\int_0^1 t^qs^mx_3(s)ds=$$

$$=\sum_{i=0}^{p}a_it^i+b\int_0^1 t^qs^m\left(\sum_{i=0}^{p}a_is^i+bs^q\sum_{i=0}^{p}\frac{a_i}{m+i+1}+\frac{b^2s^q}{m+q+1}\sum_{i=0}^{p}\frac{a_i}{m+i+1}\right)ds=$$

$$=\sum_{i=0}^{p}a_it^i+bt^q\left[\sum_{i=0}^{p}\left(a_i\int_0^1 s^{m+i}ds\right)+\sum_{i=0}^{p}\left(\frac{a_ib}{m+i+1}\int_0^1 s^{m+q}ds\right)+\frac{1}{m+q+1}\sum_{i=0}^{p}\left(\frac{a_ib^2}{m+i+1}\int_0^1 s^{m+q}ds\right)\right]=$$

$$=\sum_{i=0}^{p}a_it^i+bt^q\left[\sum_{i=0}^{p}\left(a_i\left.\frac{s^{m+i+1}}{m+i+1}\right|_0^1\right)+\sum_{i=0}^{p}\left(a_ib\left.\frac{s^{m+q+1}}{(m+i+1)(m+q+1)}\right|_0^1\right)+\sum_{i=0}^{p}\left(a_ib^2\left.\frac{s^{m+q+1}}{(m+i+1)(m+q+1)^2}\right|_0^1\right)\right]=$$

$$=\sum_{i=0}^{p}a_it^i+bt^q\sum_{i=0}^{p}\frac{a_i}{m+i+1}+b^2t^q\sum_{i=0}^{p}\frac{a_i}{(m+i+1)(m+q+1)}+b^3t^q\sum_{i=0}^{p}\frac{a_i}{(m+i+1)(m+q+1)^2}=$$

$$=\sum_{i=0}^{p}a_it^i+bt^q\sum_{i=0}^{p}\frac{a_i}{m+i+1}+\frac{b^2t^q}{m+q+1}\sum_{i=0}^{p}\frac{a_i}{m+i+1}+\frac{b^3t^q}{(m+q+1)^2}\sum_{i=0}^{p}\frac{a_i}{m+i+1}\,. \tag{140}$$

So, for $n\in\mathbb{N}\setminus\{1\}$ the $n$-th approximation

$$x_n(t)=\sum_{i=0}^{p}a_it^i+b\int_0^1 t^qs^mx_{n-1}(s)ds=$$

$$=\sum_{i=0}^{p}a_it^i+b\int_0^1 t^qs^m\left[\sum_{i=0}^{p}a_is^i+\sum_{k=1}^{n-2}\left(\frac{b^ks^q}{(m+q+1)^{k-1}}\sum_{i=0}^{p}\frac{a_i}{m+i+1}\right)\right]ds=$$

$$=\sum_{i=0}^{p}a_it^i+bt^q\sum_{i=0}^{p}\left(a_i\int_0^1 s^{m+i}ds\right)+bt^q\sum_{k=1}^{n-2}\left[\frac{1}{(m+q+1)^{k-1}}\sum_{i=0}^{p}\left(\frac{a_ib^k}{m+i+1}\int_0^1 s^{m+q}ds\right)\right]=$$

$$= \sum_{i=0}^{p} a_i t^i + bt^q \sum_{i=0}^{p} \left( a_i \left. \frac{s^{m+i+1}}{m+i+1} \right|_0^1 \right) + bt^q \sum_{k=1}^{n-2} \left[ \frac{b^k}{(m+q+1)^{k-1}} \sum_{i=0}^{p} \left( a_i \left. \frac{s^{m+q+1}}{(m+i+1)(m+q+1)} \right|_0^1 \right) \right] =$$

$$= \sum_{i=0}^{p} a_i t^i + bt^q \sum_{i=0}^{p} \frac{a_i}{m+i+1} + bt^q \sum_{k=1}^{n-2} \left[ \frac{b^k}{(m+q+1)^k} \sum_{i=0}^{p} \frac{a_i}{m+i+1} \right] =$$

$$= \sum_{i=0}^{p} a_i t^i + bt^q \sum_{i=0}^{p} \frac{a_i}{m+i+1} \left( 1 + \sum_{k=1}^{n-2} \frac{b^k}{(m+q+1)^k} \right), \qquad (141)$$

giving the geometrical progression within the last brackets. This progression is summable by $|b| < m+q+1$ and then

$$x^*(t) = \lim_{n\to\infty} x_n(t) = \lim_{n\to\infty} \left[ \sum_{i=0}^{p} a_i t^i + bt^q \sum_{i=0}^{p} \frac{a_i}{m+i+1} \left( 1 + \sum_{k=1}^{n-2} \frac{b^k}{(m+q+1)^k} \right) \right] =$$

$$= \sum_{i=0}^{p} a_i t^i + bt^q \sum_{i=0}^{p} \frac{a_i}{m+i+1} \lim_{n\to\infty} \left( 1 + \sum_{k=1}^{n-2} \frac{b^k}{(m+q+1)^k} \right) =$$

$$= \sum_{i=0}^{p} a_i t^i + bt^q \sum_{i=0}^{p} \frac{a_i}{m+i+1} \cdot \frac{1}{1 - \dfrac{b}{m+q+1}} = \sum_{i=0}^{p} a_i t^i + \frac{b(m+q+1)}{m+q+1-b} t^q \cdot \sum_{i=0}^{p} \frac{a_i}{m+i+1}. \qquad (142)$$

For avoiding triviality, the set of polynomial parameters $\{a_i\}_{i=0}^{p}$ should contain at least one nonzero element, that is $a_i \in \mathbb{R} \ \ \forall\, i = \overline{0,\, p}$ in $\exists\, i_0 \in \left\{\overline{0,\, p}\right\}$ with $a_{i_0} \in \mathbb{R} \setminus \{0\}$. The theorem, generalizing the Theorem 16, has been proved.

**Tabulating the integral equations (6) — (22) solutions**

It is easy to learn that all the proved integral equations (6) — (22) to have their unique solution $x^*(t) \in \boldsymbol{X}_{[0;\,1]}$ are contractive mappings [2, 4, 5, 8]. Those unique solutions are being tabulated as follows.

Table 1

**The exact solutions of the simplest integral equation (3) forms, having applied their being contractive mappings**

| Integral equation (3) form by $a \in \mathbb{R} \setminus \{0\}$ and $a_i \in \mathbb{R} \ \ \forall\, i = \overline{0,\, p}$ in $\exists\, i_0 \in \left\{\overline{0,\, p}\right\}$ with $a_{i_0} \in \mathbb{R} \setminus \{0\}$, $q \geqslant 0$, $m \geqslant 0$, $p \in \mathbb{N} \cup \{0\}$ | Integral equation (3) form exact solution |
|---|---|
| $x(t) = a + b\int_0^1 x(s)\,ds$ | $x^*(t) = \frac{a}{1-b}$, $\lvert b\rvert < 1$ |
| $x(t) = at + b\int_0^1 x(s)\,ds$ | $x^*(t) = at + \frac{ab}{2(1-b)}$, $\lvert b\rvert < 1$ |
| $x(t) = a + b\int_0^1 tx(s)\,ds$ | $x^*(t) = a + \frac{2ab}{2-b}t$, $\lvert b\rvert < 2$ |
| $x(t) = at + b\int_0^1 tx(s)\,ds$ | $x^*(t) = \frac{2a}{2-b}t$, $\lvert b\rvert < 2$ |
| $x(t) = a + b\int_0^1 sx(s)\,ds$ | $x^*(t) = \frac{2a}{2-b}$, $\lvert b\rvert < 2$ |

| Integral equation (3) form by $a \in \mathbb{R} \setminus \{0\}$ and $a_i \in \mathbb{R}$ $\forall\, i = \overline{0,\, p}$ in $\exists\, i_0 \in \{\overline{0,\, p}\}$ with $a_{i_0} \in \mathbb{R} \setminus \{0\}$, $q \geqslant 0$, $m \geqslant 0$, $p \in \mathbb{N} \cup \{0\}$ | Integral equation (3) form exact solution |
|---|---|
| $x(t) = at + b\int_0^1 sx(s)\,ds$ | $x^*(t) = at + \frac{2ab}{3(2-b)}$, $\lvert b \rvert < 2$ |
| $x(t) = a + b\int_0^1 tsx(s)\,ds$ | $x^*(t) = a + \frac{3ab}{2(3-b)}t$, $\lvert b \rvert < 3$ |
| $x(t) = at + b\int_0^1 tsx(s)\,ds$ | $x^*(t) = \frac{3a}{3-b}t$, $\lvert b \rvert < 3$ |
| $x(t) = a + b\int_0^1 t^q x(s)\,ds$ | $x^*(t) = a + \frac{ab(q+1)}{q+1-b}t^q$, $\lvert b \rvert < q+1$ |
| $x(t) = at + b\int_0^1 t^q x(s)\,ds$ | $x^*(t) = at + \frac{ab(q+1)}{2(q+1-b)}t^q$, $\lvert b \rvert < q+1$ |
| $x(t) = a + b\int_0^1 s^m x(s)\,ds$ | $x^*(t) = \frac{a(m+1)}{m+1-b}$, $\lvert b \rvert < m+1$ |
| $x(t) = at + b\int_0^1 s^m x(s)\,ds$ | $x^*(t) = at + \frac{ab(m+1)}{(m+2)(m+1-b)}$, $\lvert b \rvert < m+1$ |
| $x(t) = a + b\int_0^1 ts^m x(s)\,ds$ | $x^*(t) = a + \frac{ab(m+2)}{(m+1)(m+2-b)}t$, $\lvert b \rvert < m+2$ |
| $x(t) = at + b\int_0^1 ts^m x(s)\,ds$ | $x^*(t) = \frac{a(m+2)}{m+2-b}t$, $\lvert b \rvert < m+2$ |
| $x(t) = a + b\int_0^1 t^q s^m x(s)\,ds$ | $x^*(t) = a + \frac{ab(m+q+1)}{(m+1)(m+q+1-b)}t^q$, $\lvert b \rvert < m+q+1$ |
| $x(t) = at + b\int_0^1 t^q s^m x(s)\,ds$ | $x^*(t) = at + \frac{ab(m+q+1)}{(m+2)(m+q+1-b)}t^q$, $\lvert b \rvert < m+q+1$ |
| $x(t) = \sum_{i=0}^{p} a_i t^i + b\int_0^1 t^q s^m x(s)\,ds$ | $x^*(t) = \sum_{i=0}^{p} a_i t^i + \frac{b(m+q+1)}{m+q+1-b}t^q \cdot \sum_{i=0}^{p} \frac{a_i}{m+i+1}$, $\lvert b \rvert < m+q+1$ |

Speaking strictly, the solutions (23), (31), (38), (45), (52), (59), (66), (73), (80), (87), (94), (101), (108), (115), (122), (129) may be driven out directly from the solution (136), tabulated in the last line of table 1. And particularizing the functional space $\boldsymbol{X}_{[0;\,1]}$ [9], containing all these solutions, as well as the other functions $x(t)$, defined on the unit segment $[0;1]$, does not matter at all, because there may be practically any functional space with $[0;1]$-defined measurable elements.

## Conclusion

Having proved the 16 simplest integral equation forms (6) — (21) to hold their unique and exact solutions $x^*(t) \in \boldsymbol{X}_{[0;\,1]}$ as (23), (31), (38), (45), (52), (59), (66), (73), (80), (87), (94), (101), (108), (115), (122), (129) correspondingly, and the generalized integral equation form (22) to hold its unique and exact solution $x^*(t) \in \boldsymbol{X}_{[0;\,1]}$ as (136), the table 1 gives the possibility to solve at once a wide range of practical problems, modeled in the investigated simplest forms of the integral equation (3), particularized the mapping (1). In proceeding the investigation, the kernel $K(t,\, s) \in \boldsymbol{X}_{[0;\,1]} \times \boldsymbol{X}_{[0;\,1]}$, taken here as $K(t,\, s) = bt^q s^m$ with $q \geqslant 0$ and $m \geqslant 0$ generally, is going to be complicated. For instance, it must be exponential $K(t,\, s) = be^{t-s}$ as many practical problems issue from.

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